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HIERARCHY MODELS OF CITY SIZE: AN ECONOMIC EVALUATION

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In this paper, the Lösch-Beckmann hierarchy model of city size is evaluated in terms of its economic content and consistency. It is shown the model violates basic economic principles. An alternative framework is outlined and for a simple example the properties of the model are examined.

1. Hierarchy Models of City Size: A Critique

A hierarchy model of cities should be consistent with the precepts of market area analysis exposited in Chapter 10 of Lösch's *The Economics of Location*. One product is produced with ubiquitous resources and marketed amongst a rural population evenly distributed across a flat featureless plain. A honeycomb network of market areas results with a firm at the center of each market area. The size of a firm and its market area are derived by trading off the benefits of production agglomeration which are declining average costs of production against agglomeration costs which are increasing transportation costs of supplying a larger rural market area.

Having rigorously developed the network of market areas for distributing one good, Lösch in Chapter 11 of *The Economics of Location* suggested a non-rigorous generalization of his previous analysis to a distribution network for multiple goods. A network of market areas for each good is postulated and the different networks are then geometrically overlapped with the constraint that they form a consistent general pattern of goods production and distribution. Beckmann in a succession of papers (1958, 1968, 1970b) refined Lösch's original approach to account for the effect of supplying the different market centers with goods, as well as the rural population. We briefly present a well-known algebraic version of the Lösch-Beckman hierarchy model found in Beckmann (1968) and then show how it violates Lösch's principles of market area analysis as well as other economic concepts.

The model is presented in figure 1. There is a fixed uniform distribution of rural population. First order centers supply themselves and their immediate rural population with good 1. If P_1 is the population of a first order center and R is its rural population,

$$P_1 = K(P_1 + R) , \qquad K < 1 \tag{1}$$

K is the fixed factor of proportionality representing the ratio of producing population needed to supply the consuming population with a good.

Figure 1 - The Hierarchy Model

Second order centers produce goods 1 and 2. They supply good 2 to their rural and first order center markets, good 1 to their immediate rural population (of the same size as a first order market), and both goods to themselves. Third order centers export good 3 to the rural population and to the first and second order market centers. They also supply goods 2 and 1 to the second and first order rural market areas immediately surrounding them and they supply all three goods to themselves.

In general Q_n, the total population of an nth order market area, equals

$$Q_n = 1/K\,P_n = S\,Q_{n-1} + Q_{n-1} - P_{n-1} + P_n \qquad (2)$$

S is the number of n-1 order satellites of an n^{th} order city. $Q_{n-1} - P_{n-1}$ is the market population of the n^{th} order center's own n-1 order rural market area and P_n is its own urban population. Combining (1) and (2) we find that

$$P_n = \frac{(1+S-K)^n}{(1-K)^n} \cdot \frac{K \cdot R}{(1+S-K)} \qquad (3)$$

From (3), and given the model, it should be obvious that city size increases as we move up the hierarchy.

The model is just an algebraic construct with no explicit structure; but, then, the underlying structure of the model would require a very complex specification. One question is whether any underlying economic structure could generate such a model. There are three reasons why it could not.

With respect to underlying supply or cost functions, from Lösch's market area analysis, we know there must be some declining average cost of production function to ensure agglomeration of production in market centers. In the Lösch-Beckmann work, declining average costs are attained by assuming initial fixed production costs and constant marginal cost. In figure 1, the quantities of goods 1 and 2 produced in the three centers increases as we move up the hierarchy and hence their average production costs should fall. For example, third order centers supply good 1 to the same rural population as first order centers but they supply a much larger urban population. If average production costs are lower for a center, then it should supply the good to a *larger* rural population and *market area* because it can absorb higher transportation costs to the rural area. This follows from the first principles of market area analysis but is ignored in the hierarchy models.

Secondly, the positioning of the towns at the center of a rural market area in figure 1 may be incorrect. This positioning is correct if the centers only export to a rural area. However if there is inter-urban trade and its associated transportation costs, it may be beneficial to move the different centers of the hierarchy closer together to reduce the cost of inter-urban trade while raising the cost of urban-rural trade. This point is emphasized later in the paper.

The third basic economic inconsistency of the model is that, in its present form, trade cannot be balanced between cities because cities only trade unilaterally down the hierarchy. For example, the highest order center receives no imports while its exports indicate it must be running a constant trade surplus! If rural exports to all urban areas were introduced into the model, there would be a set of prices and *possibly* a set of demand and supply conditions that would yield balanced trade for all participants.

2. An Alternative Approach and a Simple Example

In this section, in our discussion and through the use of a simple example, we present the basic concepts needed to formulate a sophisticated hierarchy model of city size. We suggest that future models should incorporate the recent results of residential location theory found in Alonso (1964), Muth (1969), and Mills (1967, 1969). Production of export goods in a city also entails production of housing for the labourers employed in traded good production. This in turn entails daily commuting from the homesite to the worksite in the city center and back. As city size increases spatially the average commuting distance of workers increases and so does congestion. This is obviously a rising resource cost of increasing city size; and in Mills' work (1967, 1969), this cost eventually offsets the benefits of agglomeration (falling average costs of traded good production) to limit efficient city size. Therefore there are two factors limiting efficient agglomeration -- rising average commuting costs and the rising average transportation costs of increasing the city's market area, as described by Lösch and Beckmann.

For our more immediate purposes, the Mills' work provides an excellent reason why cities should specialize in the production of just one export good. That is, n^{th} order centers would produce only export good n. Locating the production of two export goods in a city does nothing to affect production conditions, providing there are no positive production externalities between the industries.[1] Their average production costs decline with the labour force employed in *that* industry. However, average commuting costs rise with city, not industry, population. Locating the two goods in the same city, by increasing the population concentrated in one city, increases average per person commuting costs far beyond the average commuting costs if the goods located in different cities and the labour force was split in concentration.

To be weighed against this case for specialization is the transportation costs of executing trade between the specialized cities. In comparing the costs and benefits of specialization, for some goods transportation costs are high enough to outweigh the reduction in average commuting costs of

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1. Positive production externalities would be industries using a common labour force and market such as in electronics or a common intermediate input such as warehousing and transport facilities in break-in-bulk points.

locating those goods in a separate set of cities. Examples are laundering, barbering, and general retail outlets for food and clothing. For other goods such as manufactured goods the commuting cost reduction of specialization dominates the trade cost argument.

We now turn to developing a simple example illustrating the principles from Lösch's market area analysis that should underlie a hierarchy model of cities. Our example does not directly incorporate the above discussion but pays tribute to it by assuming that cities specialize in the production of their export good.

Since cities specialize, city size is no longer determined by an accumulative algebraic process as in Beckmann's work, but is determined by the production technology of the city's export good, the demand for the good, the transportation costs of the good, and the integration of this type of city into a economically consistent framework of cities. In our example, we fix consumption coefficients and the variable costs of production in the economy.[2] The network of market areas is simplified to a hierarchy of two where we are examining the optimal ratio and relative locations of first to second order centers.

In our example there are two urban goods and one agricultural good. Three thousand labourers are employed in the production of each of these goods. The production of each good is 9000 units and each labourer consumes one unit of each good. Because of fixed production and consumption coefficients, the agricultural land area, but not shape, is fixed; and it is assumed

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2. The use of K, the factor of proportionality, in Beckmann's model would require such an assumption, even where K is variable between goods as in Beckmann and McPherson (1970b).

that cities occupy a negligible amount of space relative to agriculture.

The problem is depicted in figure 2. We wish to solve for the number of first order satellites the second order city has and for the distance between the cities. Our variables are the fixed costs of initiating good 1 production (F_1) in a location and the transport costs of inter-urban and urban-rural trade determined by A, the distance between first and second order cities. We wish to minimize the sum of F_1's and transport costs.

First let us examine the transportation cost argument. Given the ratio of first to second order centers depicted in figure (2a) and (2c), what is the optimal distance (A) between the centers. The maximum efficient A occurs with the first order center at the center of its market areas; the minimum efficient A approaches zero. To solve for A in figure (2a) and (2c) we minimize transport costs, T, where

$$T = 3000[A.(t_1 + t_2) + (f(A).t_1 + f(A)'t_3) + (F(A).t_2 + F(A)'.t_3)] \qquad (4)$$

where

3000 : number of units of goods 1, 2, or 3 (the agricultural good) entering trade between any two participants. That is, a total of 6000 units of each good are exported by their producers. Thus the relative consumption weights in determining spatial locations of cities are identical.

A : the distance from center 1 to center 2.

t_i : the cost of transporting good i one distance unit.

f(A) and f(A)' : the average distance between first order centers and their rural customers and vice-versa.

F(A) and F(A)' : the average distance from second order centers to their rural customers and vice-versa.

Note that f(A)' and F(A)' are not independent but are related through A. That is, first and second order centers would compete for food from their immediate rural areas, especially as A falls and the centers come closer together. Otherwise our f and F functions are just dependent on A and are unrelated to each other.

In determining A, this minimization solution involves a trade-off between reducing A and reducing trade costs of inter-urban trade, the first term in the square brackets of (4), while increasing trade costs of rural - first order center trade, the second term in (4).

Figure 2 - The Simple Example

Given the optimal A in figures (2a) and (2c) as well as the self-evident value of A in figure (2b), how do we choose between figures (2a), (2b), and (2c) for the optimal ratio of first and second order centers. Two considerations arise. First for transportation costs, we would expect in the optimal solutions to (4) that the more first order centers the lower the cost of urban-rural trade (i.e. f(A) and f(A)' would be lower). Secondly, for fixed production costs, the more first order centers the greater the number of F_1's incurred. Formally, to choose between figures (2a), (2b) and (2c) we choose the configuration that minimizes the sum of the relevant expression in (4) and SF_1 where S is the relevant number of first order centers.

From this simple example, we can see that a sophisticated hierarchy model must determine not only the optimal ratios of different types of cities, but their optimal relative location. If the transport costs of urban-rural trade are high and of inter-urban trade are low and F_1 is small, then a quadratic nesting of markets could be feasible with first order centers near the center of their rural market area. On the other hand, if transportation costs of urban-rural trade fall and agglomeration economies in good 1 production rise, the optimal A and the optimal incurrences of F_1 would decline. [Note that in figures (2a) and (2c) if the optimal A approaches zero, the figure collapses into (2b) where SF_1, f(A), and F(A) are all independently minimized.]

Suppose further that we now consider differences in production and consumption coefficients between goods. If the ratio of rural/urban population is high then the transport costs of urban-rural trade receive a higher weight (than 3000) in expression (4) and a higher ratio of first order centers would be efficient. If the ratio of rural/urban population declines then the optimal A and SF_1 should decline. It could be hypothesized that a hundred years ago a configuration such as figure (2a) was likely. However, given the technological improvements in transportation and in manufacturing production (increasing agglomeration economies) and given the tremendous decline in rural/urban population, a configuration such as figure (2b) is more likely today.

How could one generalize this simple example to account for many urban goods and multiple second and higher order centers. A "generalization" would be complex and would require first minimizing the appropriate parallel expressions to (4) for all feasible ratios of first, second, third etc. order centers to obtain the optimal relative urban locations for each ratio. It is not clear that the optimal solutions can or will involve symmetry in spatial locations and sizes of cities. For example, figure 1 is symmetric in every sense -- all n^{th} order centers and markets are identical in size and spacing. However feasible solutions could have 2, 3, 5, etc. centers of n, n-1, n-2, etc. order with resulting differences in optimal city sizes, market areas and shapes, and spacing between cities producing the same good. For example, some n-1 and n-2 centers might locate side-by-side and others apart supplying different size rural and urban populations. To reduce the number of possible configurations, one might impose a transportation network such as a rectangular grid and constrain towns to locate at points on the grid. The final optimal ratio of centers obtained by minimizing the sum of the parallel expression to (4) plus S_iF_i (the incurrences of fixed costs) might have a third and fourth order center located side by side and a non-symmetric pattern of second and first order centers and market areas spread out across the rural area.

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